Heine Borel Theorem: A brief history

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Abstract

In this paper, we'll discuss the historical background of the famous Heine-Borel theorem in short. Also, here we'll investigate the appropriateness of the name of this theorem.

Keywords: Closed set, bounded set, compact set.

Heine-Borel Theorem and its proof

We begin with Heine-Borel theorem which, in modern terminology, states that *every closed and bounded set is compact*.

Proof: Let S be such a set and $\mathcal{G}$ be *any* open cover of S.By Lindelöf Covering Theorem (see [3]), $\mathcal{G}$ has a countable subcover, say $\mathcal{G}'$ which also covers S. If $\mathcal{G}'$ be finite, there is nothing to prove. Otherwise we can enumerate all the elements in $\mathcal{G}$' as $\{ G_1, G_2, \ldots . \}$ (say).

Now, we define a collection $\{F_n\}$ of sets as $F_n = S - \bigcup_{i=1}^{n} G_i$, for all $n \in \mathbb{N}$.

Given, S is closed. Also, for each natural number$n$, $\bigcup_{i=1}^{n} G_n$ being the union of some open sets is open. Therefore, for each $n \in \mathbb{N}$, F_n is closed. Also, for each $n \in \mathbb{N}$, F_n is bounded, as S is so. Moreover, the collection $\{F_n\}$ satisfies the following chain of inclusions: $F_1 \supseteq F_2 \supseteq \cdots F_n \supseteq F_{n+1} \supseteq \cdots$.

Now we wish to show that at least one F_n is empty. For, on the contrary, assume that each member of $\{F_n\}$ is non empty. Then by Cantor's Intersection Theorem (see [2]), $\bigcap_{n=1}^{\infty} F_n$ is non empty. Let $x \in \bigcap_{n=1}^{\infty} F_n$. Then $x \in F_n$, for all $n \in \mathbb{N}$.Then, by the construction of F_n , x is in S but in none of G_i's. This contradicts the fact that the collection of all G_i's i.e. $\mathcal{G}'$ covers S. Hence, at least one of F_n's, say F_k must be empty. This shows that $(S - \bigcup_{i=1}^{k} G_i)$ is empty, indicating that $S \subseteq \bigcup_{i=1}^{k} G_{i.}$ ……...(1).

Let $\mathcal{G}''= \{G_1, \ldots . , G_k\}$. Then $\mathcal{G}''$ is a finite subcollection of $\mathcal{G}$,and by virtue of (1), $\mathcal{G}''$ covers the whole of S.

The arbitrariness of $\mathcal{G}$ proves the theorem.

Historical Background of Heine-Borel Theorem

In English speaking countries and in Germany, this theorem is called *Heine-Borel theorem* while in France it's called *Borel-Lebesgue theorem*. We'll see in the foregoing discussion that the latter name, however, is more appropriate.

This theorem was first stated (along with a proof) by Félix Édouard Justin Émile Borel (1871-1956) in his doctoral dissertation entitled *Sur quelques points de la théorie des fonctions,* published in 1895. This first version, which can be regarded as the early development of "Measure Theory", was:

> *If on a line [segment] there are infinitely many partial intervals such that each point on the line [segment] is interior to at least one interval, then one can determine effectively a finite number of intervals, chosen from the given intervals and having the same property (i.e., every point of the line [segment] is interior to some one of them).*

It is to be noted that Borel wanted to mean 'open interval' by the term 'partial interval'.

In the same year, Pierre Cousin (1867-1933) in his paper *Sur les fonctions de n variables complexes*, stated (and also proved) the next version of this theorem as:

In the plane YOX, define a connected space S bounded by a simple or complex closed contour; if to each point of S there corresponds a circle of finite radius, then the region can be divided into a finite number of subregions such that each subregion is interior to a circle of the given set having its center in the subregion.

In fact, Cousin's theorem (he called it a lemma) was the generalization of that of Borel, because it was an extension to two dimensions. Furthermore, his proof worked for arbitrary coverings. But, unfortunately, Cousin's name was largely overlooked for this theorem.

Cousin's lemma is generally attributed to Lebesgue (Henri, 1875-1941)who was said to aware of the result in 1898 (while studying the thesis of Borel) and published his proof (wonderfully elegant) in a 1904 book (monograph) *Leҫons sur l'intégration et la recherche des fonctions primitives*on Integral (Lebesgue Integral). Lebesgue's version of this theorem was:

> *If there is a family $\Delta$of intervals such that every point of an interval (a, b), containing both a and b, is interior to at least one of the Δ, then there exists a family formed of a finite number of the intervals Δ and enjoying the same property (every point of (a, b) is interior to one of them).*

It should be noted that the concept of 'open set' did not come at the time when Borel first published this theorem. In fact, it is of great controversy that who was actually responsible to bring out the fact that every closed, bounded subset of $\mathbb{R}$ has the 'open cover Property' (sometimes called *Borel-Lebesgue Property).* Perhaps it was Maurice Réne Frechet (1878-1973), who defined 'compactness' formally.

Now, how did Heine (Heinrich Eduard, 1821-1881)'s name come to an attachment to this famous theorem? In 1900, Arthur Schoenflies *(1853-1928),* a student of Heine, claimed that Borel's theorem extended a known theorem of Heine. To quote Schoenflies (translated into English):

> *….. To give a last example, I prove the following theorem of Borel, which extends a known*

theorem of Heine: V. *If on a straight line there is an infinite sequence of intervals δ, so that every point of the interval a … b is an interior point of at least one interval δ, then there is also always a finite subset of such intervals. …..*

Schoenflies actually noted the relationship between Borel's theorem and Heine's proof (published in 1872) of uniform continuity of a function in a closed interval. But many mathematicians (including Lebesgue) argued against the attachment of Heine's name to Borel's theorem, as in 1904, G. Arendt revealed the fact that Heine's theorem on uniform continuity was first actually appeared in an 1854 lecture of Peter Gustav Lejeune Dirichlet (1805-1859), who was Heine's Professor.

Now, we come to the name of this theorem. Lebesgue offered the name 'Borel-Schoenflies Theorem', while other campaigned for the name 'Borel-Lebesgue Theorem'. Hildebrandt (see [4]) used to call it 'Borel Theorem', while Borel himself called it 'The first Fundamental Theorem of Measure Theory'. Also, it is very interesting to see that Schoenflies dropped the name 'Heine' from this theorem in 1913 edition of his 1900 book.

References

1. Andre, N.R., Engdahl, S.M., Parker, A.E., *An Analysis of the First Proof of The Heine-Borel Theorem-History*, Convergence (Published by Mathematical Association of America), August, 2013.
2. Apostol, T.M., *Mathematical Analysis*, Narosa Publishing House (Indian Student Edition, 6th printing), 1988 (p. 56).
3. Gupta, A., *Introduction to Mathematical Analysis*, Academic Publishers (1e), 1987 (p. 61)
4. Hildebrandt, T., *The Borel Theorem and its generalizations*, Bulletin of the American Mathematical Society, 1926 (p. 423-474).
5. Sundstrom, M.R., *A Pedagogical History of Compactness*, The American Mathematical Monthly, Vol. 122, No. 7, June, 2010 (p. 619-635).