A gradual historic development of open set and closed set

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Abstract

In this article the interior points, exterior points, boundary points, open sets, closed sets, and some major properties of open sets and closed sets (in R, i.e. on a straight line) will be discussed. A very brief historical development of open sets will also be revealed here.

Keywords: nbd of a point, interior points, exterior points, boundary points, open sets, limit points, closed sets, derived sets, perfect sets.

1. Preliminaries: Some Basic Definitions And Properties

Definition 1.1: neighborhood of a point (in R): Let $x \in R$. Let $\delta > 0$. Then the open interval $(x-\delta, x+\delta)$ is called the δ-neighborhood (simply, the δ-nbd) of x, and it is denoted by $N(x,\delta)$ or by $N_\delta(x)$.

Definition 1.2: Interior point of a set $(\subseteq R)$: Let $S \subseteq R$. A point $x \in S$ is called an interior point of S if $\exists \delta > 0$ such that $N(x,\delta) \subset S$.

From definition it is obvious that an interior point of a given set S is a point of S. The converse is, however, not true in general. Even, it may happen that a set has no interior point. For instance, the set $\left\{\frac{1}{n} : n \in N\right\}$ has no interior point (see [2]).

Definition 1.3: Interior of a set (in R): Given a set $S(\subseteq R)$, the interior of S is denoted by $\int S$ or S^i, and is defined to be the collection of all interior points of S.

Definition 1.4:Open Set: A set $(\subseteq R)$ is called open (in R), if every point of S is an interior point of S. Alternately, we can define an open set as a set which is identical to its own interior.

Definition 1.5: Exterior point of a set $(\subseteq R)$: Let $S \subseteq R$. A point $x \in R$ is called an exterior point of S if $\exists \delta > 0$ such that $N(x,\delta) \subset (R-S)$.

It follows from definition 1.5 that an exterior point of a set $S(\subseteq R)$ lies outside S. But, a point outside S is not necessarily an exterior point of S. For instance, $\sqrt{2} \notin Q$, the set of rational numbers, but $\sqrt{2}$ is not an exterior point of Q. In fact, Q has no exterior point (see [4]).

Definition 1.6:Boundary point of a set $(\subseteq R)$: Let $S \subseteq R$. A point $x \in R$ is called a boundary point of S if x is neither an interior point nor an exterior point of S,i.e. if each and every nbd of x contains some points in S and some points outside S.

It should be noted that a boundary point of a given non-empty set S may or may not belong to S. For instance, each point of the set $S = \left\{\frac{1}{n} : n \in N\right\}$ is a boundary point of S; furthermore, 0 is also a boundary point of S, but $0 \notin S$.

Definition 1.7:Limit point of a set $(\subseteq R)$: Let $S \subseteq R$. A point $x \in R$ is called a limit point of S, if every nbd of x contains a point of S other than x.

A limit point of a given set S may or may not be in S. For instance, the set $S = \left\{\frac{1}{n} : n \in N\right\}$ has the sole limit point 0 (see [2]). Moreover, if x be a limit point of S, then every nbd of S contains infinitely many points of S (see [4]). From this aspect, a finite set cannot have any limit point.

Definition 1.8:Derived Set: Given a set $S(\subseteq R)$, the derived set of S is denoted by S', and is defined to be the collection of all limit points of S.

Definition 1.9:Closed Set: A set $(\subseteq R)$ is called open (in R), if it contains all of its limit points, i.e. if, contrapositively, no point outside S is a limit point of S. Alternately, a set $S(\subseteq R)$ is closed if it contains its own derived set as a subset.

Though in real, the words *open* and *closed* are antonymous, here, these are not negation to each other. To be precise, if a set is open (or not) then no conclusion can be made about its closed-ness and vice-versa. For instance, there are several sets (like Q) which are neither open nor closed **(see).** On the other hand, there are some sets $(\emptyset, R)$ which are both open and closed. In fact, there is a partial connection between openness and closeness of a set which is as follows.

Theorem 1.1: A set (in R) is open if its complement (in R) is closed.

For proof, (see [2]).

Definition 1.10: Perfect set: A set $S(\subseteq R)$ is said to be perfect if every point in S is a limit point of S and no point outside S is a limit point of S.

2. Development Of Open Set: A Brief History

In 1872, George Ferdinand Ludwig Cantor (1845-1918), the 'father of set theory', referred only to a point 'interior to' an interval. A few years later, in 1879, he extended his idea to 'interior points' to a continuous point set. But he never used the general idea of an 'open set', even on a straight line.

Giuseppe Peano (1858-1932), in his book *Geometric Applications of the Infinitesimal Calculus* (1887),defined a point p to be 'interior' to a point set A (in one, two or three dimensions) as "*if there is positive number r such that all those points whose distance from p is less than r belong to A"* (see [3]). In that book, *Peano* also defined that "*a point p was said to be 'exterior to' A, if p is interior to the complement of A"* (see [3]). Finally, he defined p as a "boundary point" of A "*if p was neither interior nor exterior to A"* (see [3]). Also, he realized that "*if A contains some (not all) points of the space, then A necessarily has a boundary point, which may or may not belong to A"* (see [3]). He defined 'boundary of a set'as "*the collection of all boundary points of that set*". Then, had he wished to bring the concept of open sets, he could havedefined a set to be 'open' if it was identical to the collection of its interiors. But, in fact, he did not do so.

Later, Camille Jordan (1838-1922) defined the 'interior points' of a set E to be "*those points of E that do not belong to the derived set complement of E"*, in an 1892 article entitled *Remarques sur les integralés défines*, (see [3]). With this definition, Jordan would have been able so easily to define 'open set' as a set consisting of all and only its interior points. But, surprisingly, he felt no need for the concept of 'openness' of a set. Rather, in that time, he was much more interested in defining 'boundary points' of a set (according to him, "*those points of a set which are interior to neither of the set nor its complement*"), and in showing that "*the set of all boundary points of a set is non empty and closed*". But he did not notice that to support his claim it required both the set and its complement to be non-empty, which was realized by Peano, prior to Jordan's publication.

These ideas resembled to that of Julius Wilhelm Richard Dedekind (1831-1916), developed many years earlier in an unpublished manuscript, which was first published in Dedekind's collected work in 1931. In this brief manuscript (*General Theorems about Spaces)* he defined a 'Körper' (German word, means *body*) as "*a system of points such that for each of its points p, there is a length d for which all the points whose distance from p is less than d belong to the system*" (see [3]). Notice that Dedekind's Körper was nothing but an open set in an n-dimensional Euclidean Space. With this notion of Körper, he did define 'Grenzpunkt' (boundary point) of a set P as "*apoint that is neither in nor outside P",* and 'Begrenzung' (boundary) of P to be "*the set of all boundary points of P*" (see [3]). His final result was that "*the boundary of a Körper cannot be a Körper*" (see [3]).

> *The term 'open' was first appeared in René Louis Baire (1874-1932)'s doctoral dissertation Sur les fonctions de variables réelles in 1899, where he defined 'closed sphere S' and 'open sphere S' with the same center and radius in n-dimensional Euclidean Space. Also, he there defined: "Given any point of S', there is a sphere of positive radius having this point as center, all of whose points belong to S'. More generally, I call any set of points possessing this property an "open domain" of n dimensions" (see [3]).*

The term 'open set' was first actually defined in 1902 by Henri Léon Lebesgue (1875-1941), in his dissertation *Intégrale, longueur, aire*. He defined that "*a set on a straight line to be open (ouvert) if it did not contain any point of its boundary*". He then stated, as a consequence of his definition, that "*every point of an open set E is an interior point of E*", and that "*the complement of an open set is closed*".

In 1906, William Henry Young (1863-1942), in his book *The Theory of Sets of Points*, defined open set as "*a set points which is not closed*". Clearly this definition was incompatible with that of Lebesgue. Obviously, nowadays, what we mean by an open set does not support Young's definition at all.

It should be noted that though nowadays we study an open set using the notion of neighborhood of a point, the concept of neighborhood did arise later than the concept of open sets. The concept of 'neighborhood' was first used by David Hilbert (1862-1943) in 1902.

3. Development Of Closed Set: A Brief History

The term 'limit point' was first coined, and the concept was first published by Cantor (George, 1845-

1918). But prior to him, the concept of 'limit point' was first invented by Karl Theodor Weierstrass (1815-1897), as it occurred repeatedly, as a part of his famous theorem (nowadays known as Bolzano-Weierstrass theorem), in his unpublished lectures delivered over almost two decades (see [1]). To quote Cantor (see [3]),

> "*By a 'limit point of a point-set P' I understand a point situated on a straight line in such a way that every neighborhood of the point contains infinitely many points of P. By the 'neighborhood of a point' is meant any interval which contains the point in its interior. From this it is easy to prove that a point-set consisting of infinitely many points must have one limit point.*"

Then Cantor used his new term 'limit point' to define 'first derived set of a point-set P' as the collection of all limit points of P. He then iterated the operation of taking limit points of a set $P$for n(any natural number) number of times to obtain $P^{(n)}$, which he called 'n-th derivative of P'. In 1880, Cantor extended this idea into the transfinite by defining $P^{(\infty)}$ as the intersection of all $P^{(n)}$'s.

In 1883, Cantor defined 'perfect set' as a set equal to its first derivative.

In 1884, he first defined 'closed set' as a set that contains all of its limit points. Also, he proved that

> "*any closed set P is the derived set of some set Q*" and "*the derived set of $(A \cup B)$ is the union of the derived set of A and the derived set of B*".

In an 1892 article mentioned in Section 2, Camille Jordan (1838-1922) defined 'limit point' of a set, different from Cantor's definition, but, which, after two decades, would become a standard definition, as:

> "*a point p is said to be a limit point of a set E if for any $\varepsilon > 0$ there is some point q of Esuch that the distance between p and q is less than ε*".

He then followed Cantor to define 'derived set' which was nothing new but the 'first derived set' (in Cantor's terminology). Although the aforesaid article of Jordan was highly influenced by Cantor's earlier works, Jordan defined 'perfect set' as "*a set containing its own derived set as a subset*". This was totally different from Cantor's 'perfect set'. What was designated by Jordan as a 'perfect set', nowadays we call it nothing but a closed set.

References

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