Criticism on Definition of the Diameter of Empty Set

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Abstract

There are two different definitions of the diameter of an empty set which contradict each other. In this paper, we'll focus on each of these definitions with their merits and demerits. Moreover, we'll also discuss here the definition of distance between two sets.

Keywords: Distance from a point to a set, distance between two sets, diameter of a set.

Definition 1: ***Distance from a point to a set***: Given a non empty set S, and given a point $p(\in R)$, the distance between p and S is denoted by $d(p,S)$, and is defined as (see [10])

$$d(p,S) = inf\{|p-s|: s \in S\}.$$

If, however, $S = \emptyset$, we define $d(p,S) = \infty$.

We now wish to make some comment on the definition $d(p,\emptyset) = \infty, \forall p \in R$. We notice that $d(p,S)$ is the minimum (not actually minimum, but infimum; to explain the aforesaid notion of distance in words, we use the term "minimum" so that one can get the idea of this notion with ease) of all distance between p& each and every point in S. If now, S be empty, no such distance can be found. In that case, $d(p,S)$ being the infimum of an empty collection is ∞ (see [4]). Thus, though the definition $d(p,\emptyset) = \infty, \forall p \in R$ might seem absurd at a glance, it is consistent, and really makes sense.

Suppose, two atheists A and B played a game with the rule: whoever between them utters the maximum distance, he wins. A started the game by saying "distance between the Earth & the Moon". B then overcame A by replying "distance between the Earth & the Sun". A again overruled B by uttering "distance between the Earth and the Proxima Centauri". Perhaps this would continue endlessly. But, at that moment B thought a while, and uttered "distance from the Earth to God". A shouted with anger, "are you joking? God does not exist at all". B replied with a smile, "Yeah, I do know that. But, you know, the distance between a point &*nothing* is infinity. So, I win".

Definition 2: ***Distance between two sets***: Given two non-empty sets A and B, the distance between them is denoted by $d(A,B)$, and is defined as (see [2])

$$d(A,B) = inf\{|a-b|: a \in A, b \in B\}.$$

However, if either (or both) of A and B be empty, then the set $\{|a-b|: a \in A, b \in B\}$ being empty, as in above, we define $d(A,B) = \infty$.

Definition 3: ***Diameter of a set***: Given a non empty set S, the diameter of S is denoted by $diam(S)$ or

$d(s)$, and is defined as (see [3])

$$d(S) = ^{\{|x-y|:x,y\in A\}}.$$

For an empty set $S, d(s)$ is defined to be $-\infty$ (see [3]).

Before making criticisms on the last definition, here we discuss some major properties of diameter of a non empty set.

Property 1: Given any non-empty set $A, d(A) \geq 0$. **Equality occurs iff** A **is a singleton.**

For proof, see [9].

Property 2: $A \subseteq B \Leftrightarrow d(A) \leq d(B)$, **for any two non-empty sets** A **and** B.

For proof, see [9].

Property 3: $d(A, B) \leq d(A \cup B)$, **for any two non-empty sets** A **and** B.

This directly follows from the fact that the infimum of a non empty set cannot exceed the supremum of that set.

Property 4: For any two non-empty sets A **and** $B, d(A \cup B) \leq d(A) + d(B) + d(A, B)$.

Proof: Suppose, on the contrary,

$$d(A \cup B) > d(A) + d(B) + d(A, B) \ldots\ldots (1).$$

Since $d(A \cup B) = ^{\{|x-y|:x,y\in(A\cup B)\}}, \exists u, v \in (A \cup B)$ such that

$$|u - v| > d(A) + d(B) + d(A, B) \ldots\ldots (2).$$

We observe that u, vcannot both belong to$A$. Because, otherwise,

$$|u - v| \leq ^{\{|x-y|:x,y}\in A\} = d(A) \leq d(A) + d(B) + d(A, B),$$

which would contradict (2).

Similarly, u, v cannot both belong to B.

So, exactly one of u and v is in A while the other is in B.

Without any loss of generality, assume that $u \in A, v \in B$. Then,

$$|u - v| \geq inf\{|a - b|: a \in A, b \in B\} = d(A, B) \leq d(A) + d(B) + d(A, B),$$

which again contradicts (2).

Hence, (1) cannot be true, and consequently, the result follows.

Now, we wish to make some criticisms on the definition$d(\emptyset) = -\infty$. This definition seems quite logical, because, the supremum of an empty set is $-\infty$ (see [4]). But we face some difficulties with this definition. Here, we describe some of them. Just now, we have proved the inequality$d(A \cup B) \leq d(A) + d(B) + d(A, B) \ldots (1)$, for any two non-empty sets$A \wedge B$. Now, if we wish to have this inequality valid for any pair of sets, we note that taking $A \neq \emptyset$ but $B = \emptyset$ the left side of (1) becomes $d(A)$,a finite non negative real number, while the right side of (1) reduces to $d(A) - \infty + \infty$, which is undefined. If we set $d(\emptyset) = 0$,we observe that (1) holds good for any pair of sets$A \wedge B$. Moreover, if we wish to have property (1) valid for any arbitrary set, we notice that by taking $d(\emptyset) = -\infty$ we have $-\infty \geq 0$, which is not true. But if we set $d(\emptyset) = 0$ then property (1) also holds good for empty sets. That's why some authors take the diameter of an empty set as zero.

The authors [1], [6] explicitly take the position $d(\emptyset) = 0$.The author [10] explicitly takes the position so that $d(\emptyset) = -\infty$. While the authors [7], [8] restrict the definition of diameter for non-empty sets only.

Finally, consider the definition $d(A, B) = \infty$, when at least one of $A \wedge B$ is empty. With this definition, the inequality $d(A, B) \leq d(A \cup B)$ does not hold well if one of A or B be empty, even if we set $d(\emptyset) = 0$. To make this inequality valid for any arbitrary pair of sets, some authors take the definition $d(A, B) = 0$,if $A = \emptyset$ or $B = \emptyset$.

Conclusion

It has just been shown in this article that there is always a confusion in taking the value of $d(\emptyset)$, which may be $-\infty$ or 0. A similar controversy is in taking the value of $d(A, B)$, when either of the sets A and B becomes empty; it may be ∞ or 0. Though the definitions $d(\emptyset) = -\infty$ and $d(A, B) = \infty$ when Aor $B = \emptyset$ fit with the definitions of infimum and supremum of an empty set, to make the set identities involving the notion of distance between two sets and the diameter of a set universally valid for any arbitrary set, as the present author thinks, the definitions $d(\emptyset) = 0$ and $d(A, B) = 0$ when $A(B) = \emptyset$ should be taken without any hesitation.

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