# Bolzano-Weierstrass Theorem: A brief history

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**Abstract**

*This article discusses the historical background of classic Bolzano-Weierstrass theorem in brief. Weierstrass' proof of Bolzano's theorem, which is based on 'repeated interval subdivision', is also illustrated in this article.*

**Keywords:** Infinite set, bounded set, limit point.

## Bolzano-Weierstrass Theorem and its proof

We begin with Bolzano-Weierstrass theorem which is one of the fundamental theorems in Real (or Complex) Analysis.In modern terminology, this theorem states that *every bounded and infinite subset of* $\mathbb{R}$ *has a limit point in* $\mathbb{R}$.

Proof: Let $S$ be an infinite, bounded set. Since $S$ is bounded, it entirely lies within a closed, bounded interval, say $[-a, a]$. Now, we divide the interval $[-a, a]$ into two subintervals, viz. $[-a, 0]$ and $[0, a]$. Then at least one of these two sub intervals must contain an infinite number of points of $S$, because otherwise, $S$ would be a finite set, an absurdity. Without any loss of generality, we assume that $[0, a]$ does so.

Next, we bisect $[0, a]$ to obtain $\left[0, \frac{a}{2}\right]$ and $\left[\frac{a}{2}, 0\right]$, at least one of which, say $\left[\frac{a}{2}, 0\right]$ does contain infinitely many points of $S$. We rename this interval by $[a_1, b_1]$.

Again, we bisect this interval to have a subinterval, say $[a_1, b_2]$ of $[a_1, b_1]$ containing infinitely many points of $S$.

Continuing this process endlessly we obtain a sequence of closed and bounded intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$, each of which (except the first member) is completely contained within the preceding one. Also, each interval in this nest contains an infinite number of points. Let $\xi$ be the supremum of the collection of all left end points $\{a_n\}_{n=1}^{\infty}$ of the intervals in the nest, and $\eta$ be the infimum of the collection of all right end points $\{b_n\}_{n=1}^{\infty}$. Since the length of the $n-$th subinterval in the nest is $(b_n - a_n) = \frac{a}{2^{n-1}}$, which tends to zero as $n$ does so, $\xi$ and $\eta$ coincide. We claim that $\xi(= \eta)$ is a limit point of $S$.

Let $\varepsilon > 0$ be arbitrary. By Archimedean property of real numbers (see [1]), there is an integer $k$ for which $2^{k-2} > \frac{a}{\varepsilon}$, implying $\frac{a}{2^{k-1}} < \frac{\varepsilon}{2}$. This shows that the $k-$th interval $[a_k, b_k]$ of the nest is entirely inscribed in the nbd $N(\xi, \varepsilon)$, and hence $N(\xi, \varepsilon)$ contains an infinite number of points of $S$. The arbitrariness of $\varepsilon$ proves our claim, and consequently the theorem follows.

## Historical development of Bolzano-Weierstrass Theorem

The earliest known version of this theorem occurred in an 1865 lecture by Weierstrass (Karl, 1815-1897), a German mathematician, as (see [2]):

> *If, in a bounded part of the plane, there are infinitely many points with a given property, then there is at least one point (inside that part or on its boundary) such that in every neighborhood of this point there are infinitely many points having the given property.* [translated into English]

In 1868, he gave a new version of this theorem in terms of functions as (see [2]):

> *If a function* has *definite property infinitely often within a finite domain, then there is a point such that in any neighborhood of this point there are infinitely many points with this property.*[translated into English]

First published version of this theorem was due to another German mathematician Cantor (George, 1845-1918) in 1872, when he tried to bring up the concept of 'limit point', as (see [2]):

> *….. From this it is easy to prove that a point-set consisting of infinitely many points must have a limit point.*[translated into English]

Intriguingly, Cantor did not mention any reference to where this theorem came from, nor did he prove it. Even, he did not notice that 'boundedness' of the set is an essential condition to make the proposition true.

In 1874, Weierstrass gave his 3rd version to this theorem as (see [2]):

> *In the domain of a real magnitude $x$ let another magnitude $x'$ be defined but in such a way that it can assume infinitely many values which all lie between two definite limits. Then it can be shown that in the domain of $x$ there is at least one place such that in any neighborhood, however small, of it there are infinitely many values of $x'$.*

Also, he proved this version in $n$ −dimensional real or complex space.

In 1878, he made another form of this theorem in terms of sets more than functions as (see [2]):

> *In any discrete domain of a manifold, which contains infinitely many places, there is at least one place which is distinguished by the fact that in any neighborhood of it, however small, there occur infinitely many places of the domain.*

It is too surprising that Weierstrass used the concept of 'limit point' to establish his theorem, but he never gave it a name nor he did define it precisely. Ultimately, in 1886, he gave his final version to this theorem as (see [2]):

> *If $x$ is an unbounded variable magnitude, which—as it is said—forms a simple manifold and is represented geometrically by a straight line, and if in it another variable magnitude $x'$ is defined in such a way that the number of defined places is infinite, then there is in the domain of $x$, for which $x'$ is defined, at least one place in the neighborhood of which infinitely many defined places occur. Such a place can either belong to the defined places, or not. It the latter case it is called a 'limit place' ('Grenzstelle').*

Now, what was the contribution of Bolzano (Bernard, 1781-1848) to this theorem? Did he prove this theorem? Certainly not, because, when Weierstrass gave his earliest version of this theorem, Bolzano was no more. Actually, Weierstrass' theorem owes its essential ideas to Bolzano. In 1817, Bolzano published a theorem (which he called a lemma) which stated that (see [2])

> *If a property $M$ does not apply to all values of a variable quantity $x$, but to all those that are smaller than a certain $u$, there is always a quantity $U$ which is the greatest of those of which it can be asserted that all smaller $x$ possess the property $M$.*

The idea behind Bolzano's proof of this lemma was the 'repeated interval subdivision', which Weierstrass used to prove his theorem (A proof of Bolzano-Weierstrass theorem based on this method is given at the beginning of this article).

**References:**

1. Apostol, T.M. (2002). *Mathematical Analysis*, Narosa Publishing House (1e, 20th reprint), p. 10.
2. Moore, G.H. (2008). *The emergence of open sets, closed sets, and limit points in Analysis and Topology*, Historia Mathematica, Vol. 5, Issue 3, p. 220-241.